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SPECULAR REFLECTION OF HEAT RADIATION FROM AN ARBITRARY REFLECTOR SURFACE TO AN ARBITRARY RECEIVER SURFACE

DONALD G. BURKHARD and DAVID L. SHEALY
Department of Physics and Astronomy, University of Georgia, Athens, Georgia, U.S.A.

and

ROMAN U. SEXL
Department of Physics, University of Vienna, Vienna, Austria

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Abstract—General formulas are derived which specify the heat flux over an arbitrary receiving surface for radiation incident upon and specularly reflected from an arbitrary curved surface. The direction of the reflected ray and its intersection with the receiving area provides a transformation which maps an element of reflecting area onto the receiving area through the Jacobian determinant. Results are expressed in terms of the equations of the surfaces. The general formulas are reduced to the special case for which the reflecting area is a surface of revolution.

NOMENCLATURE

S_0, S_1, S_2, area of source, reflector and receiver, respectively;
dS_0, dS_1, dS_2, element of area of source, reflector and receiver, respectively;
r, distance from dS_0 to dS_1;
$z(x, y)$, equation of reflector surface;
r', distance from dS_1 to dS_2;
$Z(X, Y)$, equation of receiver surface;
$\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$, Cartesian unit vectors;
s_0, flux density at reflector point;
B, brightness of emitting surface;
n_0, outward unit normal to emitting surface;
$\boldsymbol{n}_1, \boldsymbol{n}$, outward unit normal to reflecting surface;
$\boldsymbol{n}_2$, outward unit normal to receiver surface;
$\boldsymbol{i}$, unit vector which specifies direction of incident radiation;
$\boldsymbol{i}'$, unit vector which specifies direction of reflected radiation;
μ, angle of incidence on reflecting surface;
γ, angle of incidence on receiver surface;
$\mathscr{E}_{in}$, energy flux incident on unit area of reflector;
$\mathscr{E}$, energy flux incident on unit area of receiver;
ρ, reflectivity of reflector;
P, $\dfrac{[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1]^{\frac{1}{2}}}{[(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}}}$: ratio of xy projection of dS_1/dS_2;
$J(x, y)$, $\partial(X, Y)/\partial(x, y)$ = Jacobian determinant connecting $dxdy$ of reflector with $dXdY$ on receiver;
$f_1(x, y)$, $i'_x(x, y)/i'_z(x, y)$: ratio of x component to z component of unit vector which specifies direction of reflected radiation;
$f_2(x, y)$, $i'_y(x, y)/i'_z(x, y)$: ratio of y component to z component of unit vector which specifies direction of reflected radiation;
n, index of refraction;
k, extinction coefficient;

σ', electrical conductivity;
κ, susceptibility;
ω, angular frequency of radiation;
ε_0, permittivity of free space, $8{\cdot}85 \times 10^{-12}$ coul^2/n · m^2;
$\rho_{\parallel}$, reflectivity for radiation with electric vector parallel to plane of incidence;
$\rho_{\perp}$, reflectivity for radiation with electric vector perpendicular to plane of incidence.

I. INTRODUCTION

AN IMPORTANT problem in radiant heat transfer is to determine the spatial distribution of radiant energy originating from a source S_0 and reflected from a surface S_1 before arriving at a receiving surface S_2 (Fig. 1). One would like to know the detailed energy distribution over S_2 expressed, for example, as contours of equal illumination. Illumination is the energy crossing a unit area per unit time. Total energy flux over S_2 can be obtained by integration.

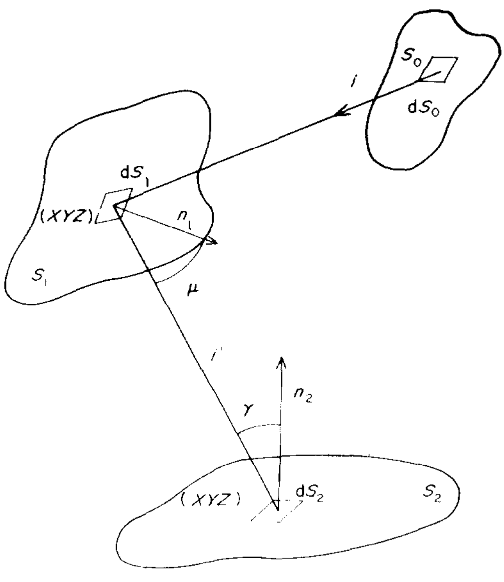

FIG. 1. Geometrical configuration reflector and receiver.

In general, part of the energy incident upon S_1 will be diffusely reflected and the remainder specularly reflected. The two contributions can be added to obtain the net heat transfer due to reflection. Diffuse reflection has been treated in detail in the literature [1]. However, no general analytical procedure has been developed for calculating the distribution over an arbitrary receiving surface for radiation which has been specularly reflected from a curved surface. One can, in principle, determine the flux by individual ray tracing but this method is cumbersome and only approximate. For example, if one ray is assigned to a unit area perpendicular to the incoming flux one can draw the reflected ray and determine its point of intersection with a receiving surface. Counting the number of rays per unit receiving area gives one a measure of the flux incident upon that area. Special cases of specular reflection have been solved analytically in the literature but the general case where S_1 is a curved surface (S_2 may also be curved) has not been solved. Specular reflection from planar surfaces has been treated in detail and also certain ruled surfaces. The usual approach is to determine the image position of S_0 created by the light reflected from S_1. The image is then regarded as a new effective source.

The image method was first applied to specular reflection from planar surfaces [2] and was later applied to cylindrical and conical cavities [3]. Somewhat later, an attempt was made [4] to generalize the image method so that one would be able to handle general non-planar reflecting surfaces; however, their approximation for the flux per unit area on a receiver surface S_2 reduces, in effect, to that which one would have obtained if the element of reflecting surface were planar. If one is to use an image method for calculating the flux per unit area on the receiver surface S_2, one must not assume as is done in [4] that an image appears at a distance from the element of reflecting surface which is equal to the distance of the source element dS_0 to the reflecting surface. Actually there will not be a simple, well defined image distance since the reflecting surface will have a variable radius of curvature depending upon the orientation of the plane formed by the normal to the surface and the incident ray [6]. An element of curved surface is characterized by two "principal" radii of curvature. The variable radius of curvature referred to (called the normal curvature) will also affect the image configuration. In general the image will be enlarged and/or compressed and distorted.

In view of the difficulties encountered when trying to apply the image method to specular reflection from nonplanar surfaces, we have developed a different and completely general method for calculating the flux density on an arbitrary receiver surface after the incident radiation has been specularly reflected from an arbitrary surface. When applying our method to planar surfaces, we obtain the same results as one would using the image method. We are also in agreement with the results of [3] for the cone or cylinder. It should be mentioned that the procedure adopted in [3] is correct for the specific ruled surfaces which were studied but the authors do not derive formulas for nor state recipes for the general problem involving arbitrary, reflecting surfaces.

When deriving formulas for the flux density on an arbitrary curved surface S_2, resulting from specular reflection of radiation from curved surface S_1, three kinds of sources of radiation may be involved. First the light incident upon S_1 may be parallel rays from infinity. Results in this case will be applicable, for example, to the calculation of energy flux over various areas of a space vehicle when sunlight is reflected from other regions of the vehicle.* Secondly the source S_0 may be a "point" source a finite distance from S_1 and S_2 with strength proportional to dS_0. Finally by integrating over all elements dS_0 one may apply the results to specular reflection of radiation originating from an extended source.

We first derive a perfectly general formula for the energy incident upon dS_2 when a given energy flux is incident upon dS_1. This phase of the calculation will be independent of the position of the source of the flux incident upon dS_1. The actual amount of flux incident upon dS_1 will vary according to the nature of the source as mentioned for the three cases above. The reflector surface will be described by the general form $z = z(x, y)$ and the receiver surface by $Z = Z(X, Y)$. Users will then be able to apply the general results to arbitrary geometries.

II. DERIVATION OF GENERAL FORMULA

The incident flux onto an element of area dS_1 on the reflecting surface is given by

$$\mathscr{E}_{in} = s_0 \cos \mu \, dS_1 \tag{1}$$

where $\cos \mu (\equiv -\boldsymbol{n}_1 \cdot \boldsymbol{i})$, $\boldsymbol{n}_1$ is the unit normal vector to the reflecting surface and $\boldsymbol{i}$ is a unit vector in the direction of the incident ray (see Fig. 1). s_0 is the energy per unit area per unit time (flux density) associated with the incident beam which in the case of parallel rays from infinity (the Sun) is the solar constant at the location of the reflecting surface. If the source emits light in accordance with Lambert's law then the flux emitted per unit area is $B\, \boldsymbol{n}_0 \cdot \boldsymbol{i}/\pi$ and s_0 in (1) is replaced by

$$s_0 \rightarrow B\, \boldsymbol{n}_0 \cdot \boldsymbol{i} \, dS_0/\pi r^2 \tag{1a}$$

where B is the brightness of dS_0 and r is the distance from dS_0 to dS_1. Then the flux per unit area on the receiver surface is equal to the flux in, $\mathscr{E}_{in}$, times the reflectivity of the reflecting surface, ρ, divided by the area out, dS_2, on the receiver surface:

$$\mathscr{E} = s_0 \rho \cos \mu \, dS_1/dS_2. \tag{2}$$

We now write an expression for dS_1/dS_2 using the basic idea that the equation for the reflected ray provides a transformation between the element of reflecting area dS_1 and the element of receiving area dS_2.

In terms of the Cartesian coordinates x, y, z on the reflecting surface and the corresponding

* The Sun subtends an angle of $\frac{1}{2}$ degree from the Earth so the rays are approximately parallel in this case.

receiving point X, Y, Z on the receiver surface, the equation of the straight line reflected ray may be written as

$$\frac{X - x}{Z(X, Y) - z(x, y)} = \frac{i'_x(x, y)}{i'_z(x, y)} \equiv f_1(x, y) \tag{3a}$$

$$\frac{Y - y}{Z(X, Y) - z(x, y)} = \frac{i'_y(x, y)}{i'_z(x, y)} \equiv f_2(x, y) \tag{3b}$$

where the equations for the reflecting surface, $z = z(x, y)$, and for the receiver surface, $Z = Z(X, Y)$, have been inserted and where (i'_x, i'_y, i'_z) specify the direction of the reflected ray. Equations (3a) and (3b), in effect, mean that we can express the element of area dS_2 in terms of the differential product $dxdy$ which can be related to dS_1. Explicitly, one has in terms of the projection $dXdY$ of dS_2 in the xy plane:

$$dS_2 = [(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}}\, dXdY \tag{4a}$$

from which

$$dS_2 = [(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}} |J(x, y)|\, dxdy \tag{4b}$$

where $J(x, y)$ is the Jacobian of the transformation (3a) and (3b) and is given by

$$J(x, y) \equiv \partial(X, Y)/\partial(x, y) \equiv (\partial X/\partial x)(\partial Y/\partial y) - (\partial X/\partial y)(\partial Y/\partial x). \tag{4c}$$

Equation (4a) is obtained by noting that the projection of dS_2 in the x,y plane is

$$dXdY = \boldsymbol{n}_2 \cdot \boldsymbol{K}\, dS_2.$$

$\boldsymbol{n}_2$ is the normal to dS_2 and is given by

$$\boldsymbol{n}_2 = \frac{\text{grad}\,[Z - Z(X, Y)]}{|\text{grad}\,[Z - Z(X, Y)]|}$$

$$= \frac{[-\boldsymbol{I}(\partial Z/\partial X) - \boldsymbol{J}(\partial Z/\partial Y) + \boldsymbol{K}]}{[(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}}}.$$

Substituting this into the above yields (4a). Equation (4b) is obtained by noting that the equations (3a, b) are nothing more than a relationship between X, Y and x, y. Thus, one can consider that the position vector of a point on the receiver surface is given by

$$\boldsymbol{R} = X(x, y)\boldsymbol{I} + Y(x, y)\boldsymbol{J} + Z(x, y)\boldsymbol{K}.$$

If one now varies x by dx holding y constant and varies y by dy holding x constant one generates two independent vectors $(\partial \boldsymbol{R}/\partial x)dx$ and $(\partial \boldsymbol{R}/\partial y)dy$ on the receiver surface whose cross product will be equal to the magnitude of the parallelogram element of area dS_2 and will be in the direction of the normal to the receiver surface, thus,

$$d\boldsymbol{S}_2 = (\partial \boldsymbol{R}/\partial x)dx \times (\partial \boldsymbol{R}/\partial y)dy$$
$$= [\boldsymbol{I}\partial(Y, Z)/\partial(x, y) + \boldsymbol{J}\partial(Z, X)/\partial(x, y) + \boldsymbol{K}\partial(X, Y)/\partial(x, y)]dxdy.$$

One may now use the general property of Jacobians

$$\frac{\partial(Y, Z)}{\partial(x, y)} = \frac{\partial(Y, Z)}{\partial(X, Y)}\frac{\partial(X, Y)}{\partial(x, y)}; \quad \frac{\partial(Z, X)}{\partial(x, y)} = \frac{\partial(Z, X)}{\partial(X, Y)}\frac{\partial(X, Y)}{\partial(x, y)}.$$

To write

$$d\boldsymbol{S}_2 = \frac{\partial(X, Y)}{\partial(x, y)}\left[-\frac{\partial Z}{\partial X}\boldsymbol{I} - \frac{\partial Z}{\partial Y}\boldsymbol{J} + \boldsymbol{K}\right] dxdy.$$

The magnitude of $d\boldsymbol{S}_2$ corresponds to (4b).

One can also express dS_1 in terms of its projection on xy plane

$$dS_1 = [(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1]^{\frac{1}{2}}\, dxdy. \tag{4d}$$

Combining the results of equations (4b) and (4d) with (2) gives a general expression for the flux per unit area on receiver surface:

$$\mathscr{E} = \frac{s_0 \rho \cos \mu[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1]^{\frac{1}{2}}}{[(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}}|J(x, y)|}. \tag{5}$$

Before equation (5) can be used for calculations, one must first express the direction cosines of the reflected ray in terms of the direction of the incident ray and the unit normal to the reflecting surface. Then equations (3a) and (b) can be used to evaluate the Jacobian as given by (4c).

The direction of the specularly reflected ray, $\boldsymbol{i}'$, must satisfy two conditions: (1) the direction of the incident ray, $\boldsymbol{i}$, the normal to the reflecting surface at the point of incidence, $\boldsymbol{n}$, and the direction of the reflected ray itself are coplanar (Law of Coplanarity) and (2) the angle between the incident ray and normal and the angle between the reflected ray and normal are equal (Law of Reflection). In general, $\boldsymbol{i}'$, can be written as a sum of components parallel to $\boldsymbol{n}$ and normal to $\boldsymbol{n}$ (see Fig. 2): $\boldsymbol{i}' = (\boldsymbol{n} \times \boldsymbol{i}) \times \boldsymbol{n} - (\boldsymbol{n} \cdot \boldsymbol{i})\boldsymbol{n}$.

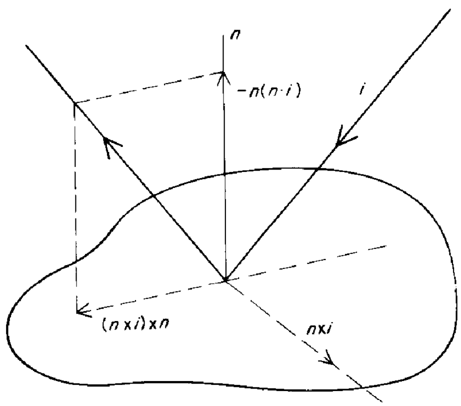

FIG. 2. Components of reflected light ray, $\boldsymbol{i}'$.

Expanding the vector triple product one has

$$\boldsymbol{i}' = -2\boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{i}) + \boldsymbol{i} \tag{6}$$

where $\boldsymbol{i}'$ is a unit vector and satisfies by construction the Law of Reflection and the Law of Coplanarity.

A general procedure for obtaining the Jacobian (4c) is to take the complete differential of (3a) and (b) where for dz and dZ one inserts the expression obtained by taking the total differential of the respective surface equations. The result will be in the form

$$\begin{aligned} A_1 \mathrm{d}X + B_1 \mathrm{d}Y &= a_1 \mathrm{d}x + b_1 \mathrm{d}y \\ A_2 \mathrm{d}X + B_2 \mathrm{d}Y &= a_2 \mathrm{d}x + b_2 \mathrm{d}y \end{aligned} \tag{7}$$

where the coefficients of dX, dY, dx, dy are then known functions of x, y, X, Y. One then solves (7) for dX and dY in terms of dx and dy:

$$\begin{aligned} \mathrm{d}X = \frac{1}{\Delta} \Bigg\{ & \Bigg[1 - \frac{\partial z}{\partial x} f_1 + (Z - z)\frac{\partial f_1}{\partial x} - \frac{\partial Z}{\partial Y} f_2 \\ & + (Z - z)\frac{\partial Z}{\partial Y}\left(\frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2\right)\Bigg] \mathrm{d}x \\ & + \Bigg[-\frac{\partial z}{\partial y} f_1 + (Z - z)\frac{\partial f_1}{\partial y} + \frac{\partial Z}{\partial Y} f_1 \\ & + (Z - z)\frac{\partial Z}{\partial Y}\left(\frac{\partial f_2}{\partial y} f_1 - \frac{\partial f_1}{\partial y} f_2\right)\Bigg] \mathrm{d}y \Bigg\} \end{aligned} \tag{7a}$$

$$\begin{aligned} \mathrm{d}Y = \frac{1}{\Delta} \Bigg\{ & \Bigg[-\frac{\partial z}{\partial x} f_2 + (Z - z)\frac{\partial f_2}{\partial x} + \frac{\partial Z}{\partial X} f_2 \\ & + (Z - z)\frac{\partial Z}{\partial X}\left(\frac{\partial f_1}{\partial x} f_2 - \frac{\partial f_2}{\partial x} f_1\right)\Bigg] \mathrm{d}x \\ & + \Bigg[1 - \frac{\partial z}{\partial y} f_2 + (Z - z)\frac{\partial f_2}{\partial y} - \frac{\partial Z}{\partial X} f_1 \\ & + (Z - z)\frac{\partial Z}{\partial X}\left(\frac{\partial f_1}{\partial y} f_2 - \frac{\partial f_2}{\partial y} f_1\right)\Bigg] \mathrm{d}y \Bigg\} \end{aligned} \tag{7b}$$

where $\Delta = 1 - \dfrac{\partial Z}{\partial X} f_1 - \dfrac{\partial Z}{\partial Y} f_2$.

Since in general

$$\begin{aligned} \mathrm{d}X &= (\partial X/\partial x)\mathrm{d}x + (\partial X/\partial y)\mathrm{d}y \\ \mathrm{d}Y &= (\partial Y/\partial x)\mathrm{d}x + (\partial Y/\partial y)\mathrm{d}y \end{aligned} \tag{8}$$

One can identify the coefficients of dx, dy in (8) with coefficients of dx, dy in the specific expressions (7a) and (b) for dX, dY to obtain explicit expressions for $(\partial X/\partial x)$, $(\partial Y/\partial x)$, $(\partial X/\partial y)$, $(\partial Y/\partial y)$. With these available one can now evaluate the Jacobian from (4c). However, if one uses this direct procedure the algebra involved to obtain a final explicit result is very lengthy.

One may considerably simplify the algebra by recognizing that equations (3a) and (b) are of the form

$$F_1(x, y, z(x, y); X, Y, Z(X, Y); f_1(x, y)) \equiv X - x - (Z - z)f_1 = 0 \quad (9a)$$

$$F_2(x, y, z(x, y); X, Y, Z(X, Y); f_2(x, y)) \equiv Y - y - (Z - z)f_2 = 0. \quad (9b)$$

Then by use of the quotient property of the Jacobian one can write

$$J(x, y) \equiv \frac{\partial(X, Y)}{\partial(x, y)} = \frac{D(F_1, F_2)/D(x, y)}{D(F_1, F_2)/D(X, Y)} \quad (10)$$

where by the chain rule for partial differentiation

$$\frac{DF_1}{Dx} = \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F_1}{\partial f_1}\frac{\partial f_1}{\partial x}$$

which makes clear the definition of D/Dx and D/Dy. The Jacobian $J(x, y)$ may be evaluated from (10) making use of the definition of the function F_1 and F_2 from (3) and (9) or one may use the brute force first method. In either case one finds

$$J(x, y) = \{I_0 + (Z - z)I_1 + (Z - z)^2 I_2\}/\Delta \quad (11)$$

where

$$I_0 = 1 - f_1(\partial z/\partial x) - f_2(\partial z/\partial y)$$

$$I_1 = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + f_1\left[\frac{\partial f_2}{\partial x}\frac{\partial z}{\partial y} - \frac{\partial f_2}{\partial y}\frac{\partial z}{\partial x}\right] - f_2\left[\frac{\partial f_1}{\partial x}\frac{\partial z}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial z}{\partial x}\right]$$

$$I_2 = (\partial f_1/\partial x)(\partial f_2/\partial y) - (\partial f_1/\partial y)(\partial f_2/\partial x)$$

$$\Delta = 1 - f_1(\partial Z/\partial X) - f_2(\partial Z/\partial Y)$$

$$f_1 \equiv i'_x/i'_z;\ f_2 \equiv i'_y/i'_z.$$

Equation (5) can then be written as

$$\mathscr{E} = \frac{s_0\rho\cos\mu[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1]^{\frac{1}{2}}|\Delta|}{[(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}}|I_0 + (Z - z)I_1 + (Z - z)^2 I_2|} \quad (12)$$

where I_0, I_1, I_2 and Δ are given by (11). We shall call equation (12) the flux flow equation.

The appearance of $Z - z$ and $(Z - z)^2$ in the denominator of (12) is due to the preferred role of z in the equation used for the surfaces. The distance r' from reflector to receiver may be displayed by noting that $(Z - z) = r'i'_z$. The term $(Z - z)^2 I_2$ $[\equiv (r')^2(i'_z)^2 I_2]$ dominates I_0 and $(Z - z)I_1$ for large values of r' and expresses the inverse square law for flux attenuation. When the radii of curvature of the reflector and receiver surfaces are comparable to r' and r, all terms are equally important. The terms I_0, I_1 and I_2 each express the role of the curvature of the reflector surface in the final expression for $\mathscr{E}$. It is possible to express equation (12) in terms of the intrinsic geometry of the surface, that is, in terms of the Gaussian curvature [6], the mean curvature and the normal curvature. When this is done and the curvature is allowed to go to zero, that is, when the element of surface is degenerated to a flat element of surface one is left with the following formula for the Jacobian (11) for point source radiation incident upon a planar facet where r' is the distance from dS_1 to dS_2. Combining (13a) with (5) or (12) using (1a) one obtains

$$J(x, y) = \frac{\boldsymbol{n}_1\cdot\boldsymbol{i}[1 + 2(r'/r) + (r'/r)^2][(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1]^{\frac{1}{2}}}{\boldsymbol{n}_2\cdot\boldsymbol{i}'[\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}}} \quad (13a)$$

$$\mathscr{E} = \frac{\rho\boldsymbol{n}_0\cdot\boldsymbol{i}\,\boldsymbol{n}_2\cdot\boldsymbol{i}'\,dS_0}{\pi(r + r')^2}. \quad (13b)$$

Equation (13b) is the result given in [4] for the view factor for radiation specularly reflected from a general curved surface. Thus one can see that the image method formulated in [4] inadvertently neglects all curvature effects. For incident plane wave, $r \to \infty$, and equation (13a) when combined with (5) gives

$$\mathscr{E} = s_0\rho\cos\gamma \quad (13c)$$

as it should. The exact expression (12), containing all terms, is relatively easy to apply.

In order to calculate the flux per unit area at a specified point on the receiver surface, one need only eliminate the intermediate reflecting coordinates x, y from (12) by applying (3a) and (b). The resulting value for the flux per unit area on the receiver surface is then a function of the direction and strength of incident radiation, the equations of the reflecting surface and the receiver surface, and the reflectivity.

To obtain contours of constant illumination over the receiver surface one must essentially invert the flux flow equation. One assigns a definite value to $\mathscr{E}/s_0$ in equation (12) for the desired contour, and then solves equation (12) for either x or y whichever is more convenient. When assigning a value to $\mathscr{E}/s_0$ one should first use equation (12) to calculate a typical value for the region of interest. Equations (12), (3a) and (b) are then used to solve for X, Y and one of the coordinates x, y while the other coordinate is treated as a parameter and assigned arbitrary values which lie on the surface. The resulting point (X, Y) on the receiver surface will define a point on the given contour. Other points on the same contour are obtained by varying either x or y and solving (3a), (b) and (12) for the corresponding X, Y. Clearly this procedure can be followed for any desired value of the contour.

III. SPECULAR REFLECTION FROM A SURFACE OF REVOLUTION ONTO AN ARBITRARY RECEIVER SURFACE

In this section we shall specialize the flux flow equation (12), to the special case where the receiver surface has axial symmetry about the z axis, in which case the equation of the reflecting surface is of the form $z = z(R)$ where $R^2 = x^2 + y^2$. The equation of the receiver surface will be left in the cartesian form $Z = Z(X, Y)$. Introducing polar coordinates

$$x = R\cos\theta;\; y = R\sin\theta;\; z = z \tag{14}$$

the partial derivative with respect to x and y appearing in (11) and (12) can be transformed into partial derivatives with respect to R and θ as follows

$$\begin{aligned}\partial/\partial x &= \cos\theta\,\partial/\partial R - (\sin\theta/R)\partial/\partial\theta\\ \partial/\partial y &= \sin\theta\,\partial/\partial R + (\cos\theta/R)\partial/\partial\theta. \end{aligned}\tag{14b}$$

The flux flow equation then becomes

$$\mathscr{E} = \frac{s_0\rho\cos\mu[(\partial z/\partial R)^2 + 1]^{\frac{1}{2}}|\Delta|}{[(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}}|I_0 + (Z - z)I_1 + (Z - z)^2 I_2|} \tag{15}$$

where $I_0 = 1 - (\partial z/\partial R)\,(f_1\cos\theta + f_2\sin\theta)$

$$\begin{aligned} I_1 &= \cos\theta\cdot\left(\frac{\partial f_1}{\partial R} + \frac{1}{R}\frac{\partial f_2}{\partial\theta}\right)\\ &\quad + \sin\theta\left(\frac{\partial f_2}{\partial R} - \frac{1}{R}\frac{\partial f_1}{\partial\theta}\right)\\ &\quad + \frac{1}{R}\frac{\partial z}{\partial R}\left(f_2\,\frac{\partial f_1}{\partial\theta} - f_2\,\frac{\partial f_2}{\partial\theta}\right)\\ I_2 &= [(\partial f_1/\partial R)(\partial f_2/\partial\theta)\\ &\quad - (\partial f_1/\partial\theta)(\partial f_2/\partial R)]/R. \end{aligned}$$

The connection between the reflecting point (R, θ, z) and the receiver point (X, Y, Z) is given by

$$\frac{X - R\cos\theta}{Z(X, Y) - Z(R)} = \frac{i_x'(R, \theta)}{i_z'(R, \theta)} \equiv f_1(R, \theta) \tag{16a}$$

$$\frac{Y - R\sin\theta}{Z(X, Y) - Z(R)} = \frac{i_y'(R, \theta)}{i_z'(R, \theta)} \equiv f_2(R, \theta). \tag{16b}$$

The interpretation and application of equations (15), (16a) and (b) is the same as described in section II.

IV. VARIATION OF COEFFICIENT OF REFLECTION WITH ANGLE OF INCIDENCE AND DIRECTION OF POLARIZATION

In general the coefficient of reflection ρ appearing in equation (12) will depend on the angle of incidence and the direction of polarization. When the polarization is parallel to the plane of incidence the reflectance is given by Fresnel's equation [5]

$$\rho_{\parallel} = \frac{[(a+b)^{\frac{1}{2}} - c]^2 + [(-a+b)^{\frac{1}{2}} - d]^2}{[(a+b)^{\frac{1}{2}} + c]^2 + [(-a+b)^{\frac{1}{2}} + d]^2} \tag{17a}$$

where μ in this case is the angle of incidence ($\cos\mu = -\boldsymbol{n}_1 \cdot \boldsymbol{i}$); and

$$a = (n^2 - k^2 - \sin^2\mu)/2$$
$$b = [(n^2 - k^2 - \sin^2\mu)^2 + 4n^2k^2]^{\frac{1}{2}}/2$$
$$c = (n^2 - k^2)\cos\mu$$
$$d = 2nk\cos\mu$$

where n is the index of refraction and k the extinction coefficient. The optical constants n and k for a metal are related to the susceptibility κ and conductivity σ' through $\varepsilon_0\kappa - i\sigma'/\omega = (n - ik)^2$. Thus $\kappa = (n^2 - k^2)/\varepsilon_0$ and $\sigma' = 2nk\omega$. ε_0 is the permittivity of free space and has the value $8{\cdot}854 \times 10^{-12}$ coul^2/n$\cdot$m^2 in the mks system. ω is the angular frequency of the light.

For polarization perpendicular to the plane of incidence:

$$\rho_{\perp} = \frac{[\cos\mu - (a+b)^{\frac{1}{2}}]^2 - a + b}{[\cos\mu + (a+b)^{\frac{1}{2}}]^2 - a + b}. \tag{17b}$$

For unpolarized incident light of net intensity s_0, the final resultant flux will be given by

$$\mathscr{E} = \frac{s_0}{2}\frac{(\rho_{\parallel} + \rho_{\perp})P\cos\mu}{|J(x,y)|} \tag{18}$$

where

$$P \equiv [(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1]^{\frac{1}{2}}/[(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 + 1]^{\frac{1}{2}}.$$

If the incident light is plane polarized with direction of polarization given by $\boldsymbol{N}$, the electric vector must be decomposed into components parallel and perpendicular to the plane of incidence. If s_0 is the intensity of the polarized light the net flux is given by

$$\mathscr{E} = \frac{s_0}{2}\frac{[\rho_{\parallel}\boldsymbol{N}\cdot\boldsymbol{n} + \rho_{\perp}\boldsymbol{N}\cdot(\boldsymbol{i}\times\boldsymbol{n})/|\boldsymbol{i}\times\boldsymbol{n}|]P\cos\mu}{|J(x,y)|}. \tag{19}$$

V. SUMMARY AND CONCLUSION

The observation has been made that the equation of the reflected ray provides a transformation between the coordinates of a point on the receiver surface and a point on the reflector surface. This transformation enables one by use of the Jacobian connecting the element of reflector area to the element of receiver area to write an explicit analytical expression for the flux per unit area incident on the receiver area as given by the "flux flow equation" (12). The general flux flow equation has been specialized to the case when the reflector area is a surface of revolution, equation (15).

It is a straightforward matter to calculate the flux onto a given receiver area from equations (12) or (15). Contours of equal heat flux on the receiver surface may be obtained by solving the flux flow equation (for constant $\mathscr{E}$) simultaneously with the equations for the reflected ray, (3a) and (b). This can always be done numerically. In some cases the inversion may be carried out completely analytically (for example, for a cone) or in other cases partially inverted. For complicated geometries it may not be a simple matter to invert the Jacobian in order to solve for the loci of points on the receiving area having a specified value for the flux density. We have succeeded in carrying out complete or partial inversion, however, for three specific geometries; the results are reported in the following paper. When inversion is not possible analytically and is complicated numerically one may simply solve for the receiver point coordinates and calculate the associated flux density for a number of arbitrary or systematically chosen incident rays. This is easy to do. It may then be possible to connect points of approximately equal flux values by hand or perhaps to least square fit a polynomial to points of approximately equal flux values. When one is only interested in total heat transfer to a given area it is only necessary to integrate the flux over the area, and contours are not necessary. The flux contours contain much more information than total flux.

It should be remarked that the general results described in this paper are not restricted to radiation *reflected* from a curved surface. By replacing equation (4) by the equation for *refracted* rays one may substitute the expressions for the refracted, x, y, z components in equations (3a) and (b) and perform similar calculations for refracted flux. This particular case usually is not of interest in heat transfer but is important in optics. All of the preceding results are applicable whenever the incident waves obey ray optics. The general criterion is that the wavelength should be short relative to the dimension of the reflecting or refracting objects. Thus the formulas developed here will be applicable to propagation of ultrasonic waves, short wavelength seismic waves, ray optics as well as radiant heat transfer.

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REFLEXION SPECULAIRE DU RAYONNEMENT THERMIQUE PAR UNE SURFACE REFLECHISSANTE ARBITRAIRE SUR UNE SURFACE RECEPTRICE ARBITRAIRE

Résumé—On établit des formules générales qui déterminent le flux thermique sur une surface réceptrice arbitraire pour un rayonnement incident et spéculairement réfléchi par une surface incurvée arbitraire. La droite support du rayon réfléchi et son intersection avec la surface réceptrice impliquent une transformation qui lie un élément de surface réfléchissante à l'aire réceptrice par le déterminant Jacobien. Les résultats sont exprimés en fonction des équations des surfaces. Les formules générales sont réduites au cas spécial où la surface réfléchissante est de révolution.

SPIEGELNDE REFLEXION DER WÄRMESTRAHLUNG VON EINER WILLKÜRLICHEN REFLEKTORFLÄCHE AUF EINE WILLKÜRLICHE EMPFÄNGERFLÄCHE

Zusammenfassung—Es werden allgemeine Gleichungen abgeleitet, die den Wärmestrahlungstrom auf eine willkürliche Empfängerfläche beschreiben, der von einer willkürlich geformten Fläche spiegelnd reflektiert wird.

Die Richtung des reflektierten Strahles und sein Durchstosspunkt mit der Empfängerfläche definieren eine Transformation, die ein Element des reflektierenden Gebietes auf das Empfängergebiet über die Jakobische Determinante abbildet. Die Ergebnisse werden in Abhängigkeit von den Gleichungen der Oberflächen angegeben. Die allgemeinen Gleichungen werden auf den Spezialfall der Rotationsfläche als reflektierendes Gebiet beschränkt.

ЗЕРКАЛЬНОЕ ОТРАЖЕНИЕ ТЕПЛОВОГО ИЗЛУЧЕНИЯ ОТ ПРОИЗВОЛЬНОЙ ПОВЕРХНОСТИ ОТРАЖАТЕЛЯ К ПРОИЗВОЛЬНОЙ ПОВЕРХНОСТИ ПРИЕМНИКА

Аннотация—Получены общие формулы для определения потока тепла на произвольной поверхности приемника излучения, падающего на искривленную поверхность произвольной формы, а затем отраженного от неё. Направление отраженного луча и его пересечение с поверхностью приемника позволяет произвести преобразование, которое отображает элемент отражающей поверхности на принимающей с помощью детерминанты Якоба. Результаты представлены уравнениями для поверхности. Получены общие формулы для случая, когда отражающая поверхность является поверхность вращения.